Solution for HW4 12-10-2016

§25) 1) c) $f(z) = e^{-y}\sin x - ie^{-y}\cos x$

So $u(x,y) = e^{-y}\sin x$ and $v(x,y) = -e^{-y}\cos x$.

Since $\begin{cases} u_x = e^{-y}\cos x = v_y \\ u_y = -e^{-y}\sin x = -v_x \end{cases}$ on $\mathbb{C}$, f is entire.

2) c) $f(z) = e^y e^{ix} = e^y\cos x + ie^y\sin x$.

So $u(x,y) = e^y\cos x$ and $v(x,y) = e^y\sin x$.

Note that CR-equation holds

$\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} -e^y\sin x = e^y\sin x \\ e^y\cos x = -e^y\cos x \end{cases} \Rightarrow \begin{cases} \sin x = 0 \\ \cos x = 0 \end{cases}$

Since $\begin{cases} \sin x = 0 \\ \cos x = 0 \end{cases} \Rightarrow \begin{cases} x = n\pi \\ x = n\pi + \frac{\pi}{2} \end{cases}$ for some $n \in \mathbb{Z}$, which is impossible, f is nowhere analytic on $\mathbb{C}$.

7) Let $f = u + iv$. Since it's analytic, we have $u_x = v_y$ & $u_y = -v_x$. Since f is real-valued, $v = 0$. Hence $u_x = 0 = u_y$ and $f = u$ must be constant.

§29) 1) a) $e^{2 \pm 3\pi i} = e^2 \cdot e^{\pm 3\pi i} = -e^2$

7) $|e^{-2z}| < 1$ iff $|e^{-2x} \cdot e^{-2yi}| < 1$ iff $|e^{-2x}| < 1$ iff $Re(z) = x > 0$

8) c) $e^{2z-1} = e^{(2x-1) + 2yi}$

So $e^{2z-1} = 1$

$\Rightarrow \begin{cases} 2x - 1 = 0 \\ 2y = 2n\pi, \ n \in \mathbb{Z} \end{cases}$

$\Rightarrow \begin{cases} x = \frac{1}{2} \\ y = n\pi, \ n \in \mathbb{Z} \end{cases}$

$\Rightarrow z = \frac{1}{2} + n\pi, \ n \in \mathbb{Z}$.

12) $e^{\frac{1}{z}} = e^{\frac{1}{x+iy}} = e^{\frac{x-iy}{x^2+y^2}} = e^{\frac{x}{x^2+y^2}}\cos\frac{y}{x^2+y^2} - ie^{\frac{x}{x^2+y^2}}\sin\frac{y}{x^2+y^2}$

$\therefore Re(e^{\frac{1}{z}}) = e^{\frac{x}{x^2+y^2}}\cos y$.

It is harmonic in every domain that does not contain the origin since it is the real part of an analytic function there.

§31) 1) a) $\mathrm{Log}(ei) = \mathrm{Log}(e^{i\pi} e^{1} \cdot e^{i\frac{\pi}{2}}) = \mathrm{Log}(e \cdot e^{i\frac{-3\pi}{2}}) = \ln(e) + i(\frac{-\pi}{2})$

$\Rightarrow \mathrm{Log}(-ei) = 1 - \frac{\pi}{2}i.$

2) b) $\log(i) = \log(e^{i\frac{\pi}{2}}) = \ln(1) + (\frac{\pi}{2} + 2n\pi)i = (2n+\frac{1}{2})\pi i, \; n \in \mathbb{Z}.$

§32) 2) For two complex numbers z_1 and z_2,

① $\ln|z_1 z_2| = \ln|z_1||z_2| = \ln|z_1| + \ln|z_2|$

② $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \in (-2\pi, 2\pi)$

Hence $\mathrm{Log}(z_1 z_2) = \ln|z_1 z_2| + i\,\mathrm{Arg}(z_1 z_2)$

$= \ln|z_1| + i\,\mathrm{Arg}(z_1) + \ln|z_2| + i\,\mathrm{Arg}(z_2) + 2N\pi i$

$= \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 + 2N\pi i, \; N = 0, 1$ or -1

§33) 1) a) $(1+i)^i = e^{i\log(1+i)}$

$= e^{i(\ln\sqrt{2} + i(\frac{\pi}{4} + 2n\pi))}$

$= e^{(-\frac{\pi}{4} + 2n\pi)} e^{i\frac{\ln 2}{2}}$

2) b) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i} = \left[e^{1-\frac{2\pi}{3}i}\right]^{3\pi i}$

$\therefore$ P.V. of $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$

$= e^{3\pi i(\ln e + i(\frac{-2\pi}{3}))}$

$= e^{2\pi^2 + 3\pi i}$

$= -e^{2\pi^2}$

8) a) $z^{c_1} \cdot z^{c_2} = e^{c_1 \mathrm{Log} z} \cdot e^{c_2 \mathrm{Log} z} = e^{(c_1+c_2)\mathrm{Log} z} = z^{c_1+c_2}$

b) $\dfrac{z^{c_1}}{z^{c_2}} = \dfrac{e^{c_1 \mathrm{Log} z}}{e^{c_2 \mathrm{Log} z}} = e^{(c_1-c_2)\mathrm{Log} z} = z^{c_1-c_2}$

c) $(z^c)^n = (e^{c \mathrm{Log} z})^n = e^{nc \mathrm{Log} z} = z^{nc}$

§34) 2) a) $e^{iz_1} \cdot e^{iz_2} = (\cos z_1 + i\sin z_1)(\cos z_2 + i \sin z_2)$

$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$

Replace z_1, z_2 by $-z_1, -z_2$, we have

$e^{-iz_1} \cdot e^{-iz_2} = \cos(-z_1)\cos(-z_2) - \sin(-z_1)\sin(-z_2)$
$+ i(\sin(-z_1)\cos(-z_2) + \cos(-z_1)\sin(-z_2))$
$= \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$
$- i(\sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2))$

b) $\sin(z_1+z_2) = \dfrac{1}{2i}(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2})$

$= \dfrac{1}{2i}(2i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2))$ (By a))

$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

14) a) $\overline{\cos(iz)} = \overline{\dfrac{e^{-z}+e^{z}}{2}} = \overline{\dfrac{e^{-x-iy}+e^{x+iy}}{2}} = \dfrac{e^{-x+iy}+e^{x-iy}}{2}$

$\Rightarrow \overline{\cos(iz)} = \cos(i\bar{z})$.

b) $\overline{\sin(iz)} = \overline{\left(\dfrac{e^{-z}-e^{z}}{2i}\right)} = \dfrac{e^{-x+iy}-e^{x-iy}}{-2i} = \dfrac{e^{x-iy}-e^{-x+iy}}{2i}$

$\sin(i\bar{z}) = \dfrac{e^{-\bar{z}}-e^{\bar{z}}}{2} = \dfrac{e^{-x+iy}-e^{x-iy}}{2i}$

$$\therefore \overline{\sin(iz)} = \sin(i\bar{z}) \iff e^{x-iy} = e^{-x+iy}$$
$$\iff e^{2x-i2y} = 1$$
$$\iff x = 0 \text{ and } 2y = 2n\pi,\ n \in \mathbb{Z}$$
$$\iff z = n\pi i,\ n \in \mathbb{Z}.$$

§35) 1) $\sinh z = \sinh x \cos y + i \cosh x \sin y$.

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So
$$\frac{d}{dz}(\sinh z) = (\sinh x \cos y)_x + i(\cosh x \sin y)_x$$
$$= \cosh x \cos y + i \sinh x \sin y$$
$$= \cosh z.$$

and
$$\frac{d}{dz}(\cosh z) = (\cosh x \cos y)_x + i(\sinh x \sin y)_x$$
$$= \sinh x \cos y + i \cosh x \sin y$$
$$= \sinh z.$$

§36) 1) a) $\tan^{-1}(2i) = \frac{i}{2} \log \frac{i+2i}{1-2i}$
$$= \frac{i}{2} \log(-3)$$
$$= \frac{i}{2} \log(3e^{i\pi})$$
$$= \frac{i}{2}\left(\ln 3 + i(\pi + 2n\pi)\right),\ n \in \mathbb{Z}$$
$$= \left(n + \frac{1}{2}\right)\pi + \frac{i}{2}\ln 3,\quad n \in \mathbb{Z}.$$